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Connection between type A and E factorizations and construction of satellite algebras

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Abstract. Recently, we introduced a new class of symmetry algebras, called satellite algebras, which connect with one another wavefunctions belonging to different potentials of a given family, and corresponding to different energy eigenvalues. Here the role of the factorization method in the construction of such algebras is investigated. A general procedure for determining an $so(2, 2)$ or $so(2, 1)$ satellite algebra for all the Hamiltonians that admit a type E factorization is proposed. Such a procedure is based on the known relationship between type A and E factorizations, combined with an algebraization similar to that used in the construction of potential algebras. It is illustrated with examples of the generalized Morse potential, the Rosen–Morse potential and the Kepler problem in a space of constant negative curvature, and, in each case, the conserved quantity is identified. It should be stressed that the method proposed is fairly general since the other factorization types may be considered as limiting cases of type A or E factorizations.

1. Introduction

Lie algebraic techniques have proved very useful in explaining the exact solvability of quantum mechanical problems [1]. Such techniques arose from the factorization method, introduced by Schrödinger [2] and later developed by Infeld and Hull [3].

More recent developments in algebraic methods, such as the introduction of the potential algebra concept [4–7] and the SUSYQM superalgebra scheme [8] for shape-invariant potentials [9], also heavily rely on the factorization method (see e.g. [7, 10]). All these approaches allow one to connect with one another wavefunctions $\psi^{(m)}(x)$ corresponding to the same energy eigenvalue, but to different potentials $V^{(m)}(x)$, $m = 0, 1, 2, \dots, n - 1$, of a given family, which may be called *satellite potentials*. In the factorization method, the ladder operators connecting $\psi^{(m)}(x)$ to $\psi^{(m+1)}(x)$ or $\psi^{(m-1)}(x)$ are m -dependent. In the potential algebra approach, this m dependence is eliminated by introducing some auxiliary variables, so that the resulting operators become the generators of some Lie algebra. The latter is compact or noncompact, according to whether n is finite or infinite. In the SUSYQM approach, in contrast, the same elimination is performed by transforming the ladder operators into supercharge ones and by introducing a supersymmetric Hamiltonian, thereby giving rise to an $su(1/1)$ superalgebra.

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In a recent paper [11], we introduced a new class of symmetry algebras, which may be called *satellite algebras*. They are similar to the potential algebras in the sense that they also depend upon some auxiliary variables and connect among themselves wavefunctions belonging to different satellite potentials. However, they are more general than the potential algebras, because the related wavefunctions correspond to different energy eigenvalues. There actually exists a conserved quantity, different from the energy, which is the eigenvalue of the algebra Casimir operator.

In the case studied in [11], which is that of the generalized Morse potential (GMP) [12] (related to the Manning–Rosen [13] or Eckart [14] potential), the conserved quantity is some combination of the potential parameters. This is an interesting property of the GMP satellite algebra, which may find application in molecular physics. It is indeed well known [15] that when analysing electromagnetic transitions between rovibrational bands in diatomic molecules, the initial and final electronic states are, in general, different and therefore give rise to different vibrational potentials, which should be taken into account in the calculation of Frank–Condon factors. It was suggested by Ley-Koo [16] that finding an algebra that both changes the potential and the vibrational state could be useful in this context. If we identify the initial and final potentials with GMP satellite ones and the initial and final vibrational states with some eigenstates of the latter, the GMP satellite algebra turns out to be a good candidate for such an algebra.

Since this shows that the new class of satellite algebras may be physically relevant, it is worth exploring it in more detail. In [11], the GMP satellite algebra $so(2, 2)$ was constructed in an indirect way by connecting the corresponding Schrödinger equation with either the Laplace equation on the hyperboloid or the Schrödinger equation for the Pöschl–Teller potential, then transferring the known $so(2, 2)$ symmetry algebra of the latter to the former. The relation between this procedure and the factorization method, although implicit, was left untouched.

The purpose of this paper is to investigate the role of the factorization method in the construction of satellite algebras. We shall devise a general procedure for determining an $so(2, 2)$ or $so(2, 1)$ satellite algebra for all the Hamiltonians that admit a type E factorization. Such a procedure is based upon the known relationship between type A and E factorizations [3], combined with an algebraization similar to that used in the construction of potential algebras [4–7]. It should be noted that our procedure is fairly general since the other factorization types (B, C, D) and F may be considered as limiting forms of A and E, respectively.

The general method proposed here will allow us to recover and generalize the results previously obtained for the GMP [11]. Various examples will be presented for illustrative purposes, but it is obvious that detailed numerical applications of each of them do not come within the scope of this paper and are left for future work.

This paper is organized as follows. The factorization method is briefly reviewed in section 2 and used in section 3 to provide a general construction method of satellite algebras. In sections 4–6, the latter is illustrated by considering the cases of the GMP, the Rosen–Morse potential, and the Kepler problem in a space of constant negative curvature, respectively. Finally, section 7 contains the conclusion.

2. The factorization method

Following Infeld and Hull [3], the linear second-order differential equation

$$\frac{d^2y}{dx^2} + r(x, m)y + \lambda y = 0 \quad (2.1)$$

where m is a non-negative integer and λ the eigenvalue to be determined, can be factorized if it can be replaced by each of the following two equations:

$$H^+(m+1)H^-(m+1)y(\lambda, m) = [\lambda - L(m+1)]y(\lambda, m) \tag{2.2}$$

$$H^-(m)H^+(m)y(\lambda, m) = [\lambda - L(m)]y(\lambda, m) \tag{2.3}$$

where

$$H^\pm(m) = \pm \frac{d}{dx} + k(x, m). \tag{2.4}$$

Here $k(x, m)$ is some function of x and of the parameter m , and $L(m)$ is an m -dependent real number.

If equation (2.1) can be factorized and $y(\lambda, m)$ is one of its solutions, then $H^-(m+1)$ and $H^+(m)$ act as ladder operators, i.e., they give rise to other solutions

$$y(\lambda, m+1) = H^-(m+1)y(\lambda, m) \quad y(\lambda, m-1) = H^+(m)y(\lambda, m) \tag{2.5}$$

corresponding to the same λ , but different m values. Moreover, the operators $H^-(m)$ and $H^+(m)$ are formally mutually adjoint.

In practical cases, it can be checked that if $y(\lambda, m)$ is a square-integrable solution of equations (2.2) and (2.3), and $L(m)$ is an increasing (resp. decreasing) function of m , then the operator $H^-(m+1)$ (resp. $H^+(m)$) yields a function $y(\lambda, m+1)$ (resp. $y(\lambda, m-1)$) that is also square integrable.

When $L(m)$ is an increasing (resp. decreasing) function of m , the problem is said to be of class I (resp. II). A necessary condition for square-integrable solutions is then that $\lambda = \lambda_l = L(l+1)$ (resp. $\lambda = \lambda_l = L(l)$), where l is an integer and $m = 0, 1, \dots, l$ (resp. $m = l, l+1, \dots$).

Square-integrable solutions that are also normalized are denoted by Y_l^m . In addition to equations (2.2) and (2.3), they satisfy the relations

$$H^-(m+1)Y_l^m = [\lambda - L(m+1)]^{1/2}Y_l^{m+1} \quad H^+(m)Y_l^m = [\lambda - L(m)]^{1/2}Y_l^{m-1}. \tag{2.6}$$

The possible factorizations types can be found by inserting equation (2.4) into equations (2.2) and (2.3), comparing the results with equation (2.1), and eliminating the function $r(x, m)$. This leads to a differential-difference equation for $k(x, m)$, which was shown by Infeld and Hull [3] to have six different nontrivial types of solutions, denoted by the letters A, B, C, D, E and F. From $k(x, m)$, it is then possible to find $r(x, m)$ and $L(m)$.

For type A and E factorizations, to be considered in the remainder of this paper, $r(x, m)$, $k(x, m)$, and $L(m)$ are given in terms of some constants a, c, d, p and q by

$$r(x, m) = \frac{a^2(m+c)(m+c+1) + d^2 + 2ad(m+c + \frac{1}{2}) \cos[a(x+p)]}{\sin^2[a(x+p)]} \tag{2.7}$$

$$k(x, m) = (m+c)a \cot[a(x+p)] + \frac{d}{\sin[a(x+p)]} \tag{2.8}$$

$$L(m) = a^2(m+c)^2 \tag{2.9}$$

and

$$r(x, m) = -\frac{m(m+1)a^2}{\sin^2[a(x+p)]} - 2aq \cot[a(x+p)] \tag{2.10}$$

$$k(x, m) = ma \cot[a(x+p)] + \frac{q}{m} \tag{2.11}$$

$$L(m) = a^2m^2 - \frac{q^2}{m^2} \tag{2.12}$$

respectively.

3. General construction method of satellite algebras

Let us consider the most general second-order differential equation admitting a type E factorization. From equations (2.1) and (2.10), it is given by

$$\frac{d^2\psi}{dx^2} - \left\{ \frac{m(m+1)a^2}{\sin^2[a(x+p)]} + 2aq \cot[a(x+p)] \right\} \psi + \lambda\psi = 0 \quad (3.1)$$

where the normalized eigenfunctions Y_l^m corresponding to the discrete eigenvalues $\lambda = \lambda_l$ are denoted by ψ , a , p , q are some constants, and l , m run over some non-negative integers. Equation (3.1) can be factorized as shown in equations (2.2) and (2.3), with $y(\lambda, m)$ replaced by $\psi = Y_l^m$, and $k(x, m)$, $L(m)$ given in equations (2.11) and (2.12), respectively. For definiteness sake, in the following we shall restrict ourselves to class I problems, but the results obtained can easily be accommodated to class II problems by replacing $l+1$ by l , and changing the m range accordingly.

In the ladder operator definition given in equations (2.4) and (2.11), m occurs in the denominator, so that an algebraization along the lines of [4–7] is not possible. To carry out such an algebraization, it is necessary to first transform the type E factorizable equation (3.1) into a type A one, which according to equations (2.1) and (2.7) is given by

$$\frac{d^2\chi}{dy^2} - \frac{\bar{a}^2(\bar{m} + \bar{c})(\bar{m} + \bar{c} + 1) + \bar{d}^2 + 2\bar{a}\bar{d}(\bar{m} + \bar{c} + \frac{1}{2}) \cos[\bar{a}(y + \bar{p})]}{\sin^2[\bar{a}(y + \bar{p})]} \chi + \bar{\lambda}\chi = 0. \quad (3.2)$$

Here the variable x is changed into y , a bar is put on top of all the constants to distinguish them from those used for type E factorization, and the normalized eigenfunctions \bar{Y}_l^m , corresponding to the eigenvalues $\bar{\lambda} = \bar{\lambda}_l$, are denoted by χ . From equations (2.4), (2.8) and (2.9), it follows that the associated ladder operators $\bar{H}^\pm(\bar{m})$, which depend linearly on \bar{m} , and the real constant $\bar{L}(\bar{m})$ can be written as

$$\bar{H}^\pm(\bar{m}) = \pm \frac{d}{dy} + (\bar{m} + \bar{c})\bar{a} \cot[\bar{a}(y + \bar{p})] + \frac{\bar{d}}{\sin[\bar{a}(y + \bar{p})]} \quad (3.3)$$

and

$$\bar{L}(\bar{m}) = \bar{a}^2(\bar{m} + \bar{c})^2 \quad (3.4)$$

respectively.

By performing two successive changes of variable and of function,

$$\begin{aligned} z &= \ln[\tan(a\bar{x}/2)] & \bar{x} &\equiv x + p \\ \psi(x) &= [\sin(a\bar{x})]^{1/2} \phi(z) = (\cosh z)^{-1/2} \phi(z) \end{aligned} \quad (3.5)$$

and

$$y = iz + \frac{\pi}{2} \quad \phi(z) = \chi(y) \quad (3.6)$$

equation (3.1) is transformed into an equation of type (3.2),

$$\frac{d^2\chi}{dy^2} - \frac{\frac{\lambda}{a^2} - \frac{1}{4} + 2i\frac{q}{a} \cos y}{\sin^2 y} \chi + \left(m + \frac{1}{2}\right)^2 \chi = 0. \quad (3.7)$$

Comparison between equations (3.2) and (3.7) shows that the type A factorization constants \bar{a} , \bar{c} , \bar{d} , \bar{p} , and parameter \bar{m} are connected with the constants a , q , and the eigenvalue λ of type E factorization through the relations

$$\bar{a} = 1 \quad \bar{p} = 0 \quad (3.8)$$

$$\bar{d} \left(\bar{m} + \bar{c} + \frac{1}{2} \right) = i \frac{q}{a} \quad (3.9)$$

$$(\bar{m} + \bar{c})(\bar{m} + \bar{c} + 1) + \bar{d}^2 = \frac{\lambda}{a^2} - \frac{1}{4}. \quad (3.10)$$

From equation (3.9), we get

$$\bar{m} + \bar{c} + \frac{1}{2} = \frac{iq}{a\bar{d}} \tag{3.11}$$

and by substituting this expression into equation (3.10), the latter becomes

$$-\frac{q^2}{\bar{d}^2} + a^2\bar{d}^2 = \lambda. \tag{3.12}$$

We know however that, for a type E factorizable problem of class I, the eigenvalue λ is given by

$$\lambda = L(l + 1) = a^2(l + 1)^2 - \frac{q^2}{(l + 1)^2} \tag{3.13}$$

where equation (2.12) has been used. By equating the two expressions (3.12) and (3.13) for λ , we obtain a quadratic equation for \bar{d}^2 with two real solutions, $\bar{d}^2 = (l + 1)^2$ and $\bar{d}^2 = -q^2/[a^2(l + 1)^2]$. By using equation (3.11) again, we therefore get four possible choices for \bar{d} and $\bar{m} + \bar{c} + (1/2)$:

$$\bar{d} = \epsilon(l + 1) \quad \bar{m} + \bar{c} + \frac{1}{2} = \frac{i\epsilon q}{a(l + 1)} \tag{3.14}$$

and

$$\bar{d} = \frac{i\epsilon q}{a(l + 1)} \quad \bar{m} + \bar{c} + \frac{1}{2} = \epsilon(l + 1) \tag{3.15}$$

where $\epsilon = \pm 1$ is a so far undetermined sign.

After inverting the transformations (3.5) and (3.6), and taking equation (3.8) into account, the type A ladder operators (3.3) lead to ladder operators for the original eigenfunctions ψ ,

$$\begin{aligned} \tilde{H}^\pm(\bar{m}) &\equiv [\sin(a\bar{x})]^{1/2} \tilde{H}^\pm(\bar{m}) [\sin(a\bar{x})]^{-1/2} \\ &= \mp i \frac{\sin(a\bar{x})}{a} \frac{d}{d\bar{x}} + i \left(\bar{m} + \bar{c} \pm \frac{1}{2} \right) \cos(a\bar{x}) + \bar{d} \sin(a\bar{x}) \end{aligned} \tag{3.16}$$

where any of the two substitutions defined in equations (3.14) and (3.15) may in principle be performed. We shall denote the resulting operators by $\tilde{H}_1^\pm(\bar{m})$ and $\tilde{H}_2^\pm(\bar{m})$, respectively.

Such ladder operators can now be transformed into Lie algebra generators by introducing two auxiliary variables $\xi, \eta \in [0, 2\pi)$, and extended eigenfunctions defined by

$$\Psi_{s,t}(x, \xi, \eta) = (2\pi)^{-1} e^{is\xi} \psi(x) e^{i\eta} \tag{3.17}$$

where

$$s \equiv \frac{i\epsilon q}{a(l + 1)} \quad t \equiv \epsilon(l + 1). \tag{3.18}$$

Since

$$S_0 = -i \frac{\partial}{\partial \xi} \quad T_0 = -i \frac{\partial}{\partial \eta} \tag{3.19}$$

are such that

$$S_0 \Psi_{s,t} = s \Psi_{s,t} \quad T_0 \Psi_{s,t} = t \Psi_{s,t} \tag{3.20}$$

we may replace s and t by $-i\partial/\partial\xi$ and $-i\partial/\partial\eta$ when such operators act on the extended eigenfunctions, respectively. By combining the transformations

$$(-i)e^{\pm i\xi} \tilde{H}_1^\mp(\bar{m} + \frac{1}{2} \pm \frac{1}{2}) \rightarrow S_\pm \quad (-i)e^{\pm i\eta} \tilde{H}_2^\mp(\bar{m} + \frac{1}{2} \pm \frac{1}{2}) \rightarrow T_\pm \tag{3.21}$$

with these substitutions, we obtain

$$S_{\pm} = e^{\pm i\xi} \left[\pm \frac{\sin(a\bar{x})}{a} \frac{\partial}{\partial \bar{x}} - i \cos(a\bar{x}) \frac{\partial}{\partial \xi} - \sin(a\bar{x}) \frac{\partial}{\partial \eta} \right] \quad (3.22)$$

$$T_{\pm} = e^{\pm i\eta} \left[\pm \frac{\sin(a\bar{x})}{a} \frac{\partial}{\partial \bar{x}} - i \cos(a\bar{x}) \frac{\partial}{\partial \eta} - \sin(a\bar{x}) \frac{\partial}{\partial \xi} \right]. \quad (3.23)$$

We note that S_0 , S_{\pm} and T_0 , T_{\pm} only differ by the substitutions $\xi \leftrightarrow \eta$, $\partial/\partial\xi \leftrightarrow \partial/\partial\eta$.

It is now straightforward to check that each set of generators S_0 , S_+ , S_- and T_0 , T_+ , T_- satisfies the defining relations of $su(1, 1) \simeq so(2, 1)$, e.g.

$$[S_0, S_{\pm}] = \pm S_{\pm} \quad [S_+, S_-] = -2S_0, \quad (3.24)$$

and that any generator of the first set commutes with any generator of the second one. Hence, the six operators generate an $so(2, 2) \simeq su(1, 1) \oplus su(1, 1)$ Lie algebra.

Both Casimir operators

$$C_s \equiv -S_+S_- + S_0(S_0 - 1) \quad C_t \equiv -T_+T_- + T_0(T_0 - 1) \quad (3.25)$$

are equal and given by

$$C_s = C_t = C = \sin^2(a\bar{x}) \left[\frac{1}{a^2} \frac{\partial^2}{\partial \bar{x}^2} - \frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \eta^2} - 2i \cot(a\bar{x}) \frac{\partial^2}{\partial \xi \partial \eta} \right]. \quad (3.26)$$

Since from equations (3.13) and (3.18),

$$s^2 + t^2 = \frac{\lambda}{a^2} \quad st = \frac{iq}{a} \quad (3.27)$$

the action of C on the extended eigenfunctions (3.17) is given by

$$\begin{aligned} C\Psi_{s,t}(x, \xi, \eta) &= (2\pi)^{-1} e^{i(s\xi+t\eta)} \frac{\sin^2(a\bar{x})}{a^2} \left[\frac{d^2}{d\bar{x}^2} - 2aq \cot(a\bar{x}) + \lambda \right] \psi(x) \\ &= m(m+1)\Psi_{s,t}(x, \xi, \eta) \end{aligned} \quad (3.28)$$

where in the last step use has been made of equation (3.1).

All the arguments presented so far have been rather formal. For completeness sake, we also have to discuss the eigenfunction normalizability conditions, which are known to play an important role in applying the factorization method. This will be considered for some examples in the next sections. At this stage, however, we may already note three important properties at the Lie algebra representation level.

Firstly, it is clear that such representations will be nonunitary. The lack of unitarity actually comes from the normalization change implied by the transformation from $\psi(x)$ to $\phi(z)$ in equation (3.5).

Secondly, in practice we shall have to distinguish between trigonometric and hyperbolic potentials, for which a in equation (3.1) is real or imaginary, respectively. In the former case, s , defined in equation (3.18), turns out to be imaginary. This is incompatible with the eigenvalues of S_0 in an $su(1, 1)$ irreducible representation, which should differ from one another by some real integer [17]. Hence, we are only left with the $su(1, 1)$ algebra generated by T_0 , T_+ , and T_- . In the latter case, on the contrary, by setting $a = i\alpha$ (α real), we find that

$$s = \frac{\epsilon q}{\alpha(l+1)} \quad (3.29)$$

so that both $su(1, 1)$ algebras may be considered. We shall concentrate on this case in the remainder of this paper, and therefore replace equations (3.1), (3.22) and (3.23) by

$$\frac{d^2\psi}{d\bar{x}^2} - \left[\frac{m(m+1)\alpha^2}{\sinh^2(\alpha\bar{x})} + 2\alpha q \coth(\alpha\bar{x}) \right] \psi + \lambda\psi = 0 \quad (\bar{x} \equiv x + p) \quad (3.30)$$

$$S_{\pm} = e^{\pm i\xi} \left[\pm \frac{\sinh(\alpha\bar{x})}{\alpha} \frac{\partial}{\partial\bar{x}} - i \cosh(\alpha\bar{x}) \frac{\partial}{\partial\xi} - i \sinh(\alpha\bar{x}) \frac{\partial}{\partial\eta} \right] \tag{3.31}$$

$$T_{\pm} = e^{\pm i\eta} \left[\pm \frac{\sinh(\alpha\bar{x})}{\alpha} \frac{\partial}{\partial\bar{x}} - i \cosh(\alpha\bar{x}) \frac{\partial}{\partial\eta} - i \sinh(\alpha\bar{x}) \frac{\partial}{\partial\xi} \right] \tag{3.32}$$

respectively.

Thirdly, from equation (3.28), we note that the $so(2, 2)$ irreps may be characterized by m , so that their basis functions may be denoted by $\Psi_{s,t}^{(m)}(x, \xi, \eta)$. When acting on such functions, the generators S_{\pm} of the first $su(1, 1)$ algebra change s into $s \pm 1$, while leaving t unchanged. In other words, the energy eigenvalue label l and the potential parameter m do not change, but the other potential parameter q becomes $q' = q \pm \epsilon\alpha(l + 1)$. When considering instead the generators T_{\pm} of the second $su(1, 1)$ algebra, s is left unchanged, while t changes into $t \pm 1$. In this case, the potential parameter m is still unchanged, but both l and q are changed into $l' = l \pm \epsilon$ and $q' = q(l + 1 \pm \epsilon)/(l + 1)$, respectively. It is therefore clear that the $so(2, 2)$ generators connect among themselves eigenfunctions belonging to different satellite potentials and different energy eigenvalues. We conclude that any family of type E factorizable Hamiltonians corresponding to hyperbolic potentials has an $so(2, 2)$ satellite algebra.

It should be noted that there remains an undetermined sign ϵ in the definitions (3.18) and (3.29) of s and t . In all the examples considered in the next sections, we have checked that, apart from some irrelevant phase factors, the results are independent of the choice made for ϵ . Hence, in the remainder of this paper, we shall use the convention

$$\epsilon = \frac{q}{|q|} \tag{3.33}$$

which provides the simplest link with the GMP analysis in [11].

4. The generalized Morse potential

As a first example, let us consider the GMP studied in [11, 12]. The corresponding Schrödinger equation is

$$-\frac{\hbar^2}{2\mu} \frac{d^2\psi}{dr^2} + D \left(1 - \frac{b}{e^{ar} - 1} \right)^2 \psi - E\psi = 0 \quad b = e^{ar_e} - 1 \tag{4.1}$$

where $0 \leq r < \infty$, and D, b, a are some parameters regulating the depth, position of the minimum r_e , and radius of the potential.

In terms of the parameters

$$\alpha_n = \sqrt{k - \epsilon_n} \quad \beta_n = \sqrt{\alpha_n^2 + kb(b+2)} \quad m = \frac{1}{2} \left(-1 + \sqrt{1 + 4kb^2} \right) \tag{4.2}$$

where

$$k = \frac{2\mu D}{a^2\hbar^2} \quad \epsilon_n = \frac{2\mu E_n}{a^2\hbar^2} \tag{4.3}$$

the energy eigenvalues and corresponding eigenfunctions are given by

$$E_n = D - \frac{a^2\hbar^2}{8\mu} \left(n + m + 1 - \frac{kb(b+2)}{n+m+1} \right)^2 \tag{4.4}$$

and

$$\psi_n(r) = N_n y^{\alpha_n} (1+y)^{-\beta_n} {}_2F_1(-n, -n-2m-1; 2\alpha_n+1; -y) \quad y \equiv (e^{ar} - 1)^{-1} \tag{4.5}$$

where $n = 0, 1, \dots, n_{\max}$, n_{\max} is the largest integer smaller than $\sqrt{kb(b+2)} - m - 1$, and N_n is some normalization coefficient.

Equation (4.1) can be rewritten as a type E factorizable Hamiltonian corresponding to a hyperbolic potential. By performing the change of variable $\bar{x} = ar/2$, it indeed reduces to equation (3.30), where

$$\alpha = 1 \quad q = -kb(b+2) \quad \lambda = 4(\epsilon - k) - 2kb(b+2) \quad (4.6)$$

and m is given by equation (4.2). The corresponding $L(m) = -m^2 - (q^2/m^2)$ is an increasing function of m , so that the GMP problem is of class I. Comparing then $\lambda = \lambda_l = L(l+1) = -(l+1)^2 - q^2/(l+1)^2$ with the expression for λ resulting from equations (4.3), (4.4) and (4.6), we obtain the relation

$$l = n + m \quad (4.7)$$

between the eigenvalue labels n and l , coming from the resolution of the Schrödinger equation and the factorization method, respectively.

From such a relation, we find that s and t , defined in equations (3.18), (3.29) and (3.33), become

$$s = \frac{kb(b+2)}{n+m+1} = \alpha_n + \beta_n \quad t = -n - m - 1 = \alpha_n - \beta_n \quad (4.8)$$

and therefore correspond to the quantum numbers m and g of [11], respectively[†]. Moreover, when rewritten in terms of the variables y , ξ , η (see equation (4.5)), the $so(2, 2)$ generators S_{\pm} , T_{\pm} of equations (3.31) and (3.32) are transformed into

$$S_{\pm} = e^{\pm i\xi} \left(\mp \sqrt{y(y+1)} \frac{\partial}{\partial y} - i \frac{2y+1}{2\sqrt{y(y+1)}} \frac{\partial}{\partial \xi} - i \frac{1}{2\sqrt{y(y+1)}} \frac{\partial}{\partial \eta} \right) \quad (4.9)$$

$$T_{\pm} = e^{\pm i\eta} \left(\mp \sqrt{y(y+1)} \frac{\partial}{\partial y} - i \frac{2y+1}{2\sqrt{y(y+1)}} \frac{\partial}{\partial \eta} - i \frac{1}{2\sqrt{y(y+1)}} \frac{\partial}{\partial \xi} \right) \quad (4.10)$$

and therefore coincide with the operators M^{\pm} and $-G^{\pm}$ of [11]. From equation (4.2), it follows that the conserved quantity, given by the eigenvalue m of the Casimir operator C , is here a combination of the potential parameters kb^2 , or Db^2/a^2 . The operators S_{\pm} change b into $b' = 2(kb^2)b/(2kb^2 \mp tb)$ while leaving n unchanged, whereas the operators T_{\pm} change both b and n into $b' = 2tb/(2t \pm b \pm 2)$ and $n \mp 1$, respectively. For both types of operators, k becomes $k' = kb^2/b'^2$.

We conclude that the results obtained in [11], using some *ad hoc* arguments, are but special cases of the general formalism developed in the present paper. In the next two sections, we shall prove that other examples can be treated in a similar way.

5. The Rosen–Morse potential

The Schrödinger equation for the Rosen–Morse potential [18] is

$$-\frac{\hbar^2}{2\mu} \frac{d^2\psi}{dx^2} + [B \tanh(\alpha x) - C \operatorname{sech}^2(\alpha x)]\psi - E\psi = 0 \quad (5.1)$$

where $-\infty < x < +\infty$, α determines the radius of the potential, while B , C regulate the position of its minimum $x_0 = -\alpha^{-1} \tanh^{-1}(B/2C)$ and its depth at the minimum $V(x_0) = -C - B^2/(4C)$, and are restricted by the condition $|B| < 2C$.

[†] Note that the symbols l and m of [11] correspond to $m+1$ and s in this paper, respectively.

In terms of the parameters

$$a_n = -\frac{\beta}{2b_n} \quad b_n = \sqrt{\gamma + \frac{1}{4}} - n - \frac{1}{2} \quad m = \frac{1}{2} \left(-1 + \sqrt{1 + 4\gamma} \right) \quad (5.2)$$

where

$$\beta = \frac{2\mu B}{\hbar^2 \alpha^2} \quad \gamma = \frac{2\mu C}{\hbar^2 \alpha^2} \quad (5.3)$$

the energy eigenvalues and corresponding eigenfunctions are given by [18]

$$E_n = -\frac{\hbar^2 \alpha^2}{2\mu} (a_n^2 + b_n^2) \quad (5.4)$$

and

$$\begin{aligned} \psi_n(x) &= N_n e^{a_n \alpha x} [\cosh(\alpha x)]^{-b_n} {}_2F_1(-n, 2m - n + 1; a_n + b_n + 1; y) \\ y &\equiv \frac{1}{2} [1 + \tanh(\alpha x)] \end{aligned} \quad (5.5)$$

where $n = 0, 1, \dots, n_{\max}$, n_{\max} is the largest integer smaller than $m - \sqrt{|\beta|/2}$, and N_n is some normalization coefficient [19].

Equation (5.1) can be rewritten in the form (3.30) by setting

$$p = i \frac{\pi}{2\alpha} \quad q = \frac{1}{2} \alpha \beta \quad \lambda = \frac{2\mu E}{\hbar^2} \quad (5.6)$$

while m is given by equation (5.2). The corresponding $L(m) = -\alpha^2 m^2 - \alpha^2 \beta^2 / (4m^2)$ is a decreasing function of m . The Rosen–Morse problem is therefore of class II. Comparing $\lambda = \lambda_l = L(l) = -\alpha^2 l^2 - \alpha^2 \beta^2 / (4l^2)$ with the expression for λ resulting from equations (5.2), (5.4) and (5.6), we obtain the relation

$$l = m - n \quad (5.7)$$

between the eigenvalue labels n and l , coming from the resolution of the Schrödinger equation and the factorization method, respectively.

Taking equation (5.6) into account, the operators S_{\pm} of equation (3.31) become

$$S_{\pm} = e^{\pm i\xi} \left[\pm i \frac{\cosh(\alpha x)}{\alpha} \frac{\partial}{\partial x} + \sinh(\alpha x) \frac{\partial}{\partial \xi} + \cosh(\alpha x) \frac{\partial}{\partial \eta} \right] \quad (5.8)$$

and the operators T_{\pm} are obtained from them by the transformations $\xi \leftrightarrow \eta, \partial/\partial \xi \leftrightarrow \partial/\partial \eta$.

From equations (3.18), (3.29) and (3.33) (where $l + 1$ is replaced by l as we have here a class II problem), we get

$$s = \frac{|q|}{\alpha l} = \frac{|\beta|}{2l} = -\epsilon a_n \quad t = \epsilon l = \epsilon(m - n) = \epsilon b_n \quad (5.9)$$

with $\epsilon = \beta/|\beta| = B/|B|$. By using equation (5.5) and the results of [19], the corresponding extended eigenfunctions can be written as

$$\begin{aligned} \Psi_{s,t}^{(m,\epsilon)}(x, \xi, \eta) &= (2\pi)^{-1} N_{s,t}^{(m,\epsilon)} e^{i(s\xi+t\eta)} e^{-\epsilon s \alpha x} [\cosh(\alpha x)]^{-\epsilon t} \\ &\quad \times {}_2F_1(\epsilon t - m, \epsilon t + m + 1; \epsilon(t - s) + 1; y) \\ y &\equiv \frac{1}{2} [1 + \tanh(\alpha x)] \end{aligned} \quad (5.10)$$

where

$$N_{s,t}^{(m,\epsilon)} = \frac{1}{2^{\epsilon t} \Gamma(\epsilon(t - s) + 1)} \left(\frac{\epsilon \alpha (t - s)(t + s) \Gamma(m + \epsilon t + 1) \Gamma(m - \epsilon s + 1)}{t \Gamma(m - \epsilon t + 1) \Gamma(m + \epsilon s + 1)} \right)^{1/2}. \quad (5.11)$$

After some calculations using well known properties of the hypergeometric function [20], we obtain

$$S_{\pm} \Psi_{s,t}^{(m,\epsilon)} = \epsilon i \left(\frac{(t-s)(t+s)(m \mp s)(m \pm s + 1)}{(t-s \mp 1)(t+s \pm 1)} \right)^{1/2} \Psi_{s \pm 1, t}^{(m,\epsilon)} \quad (5.12)$$

$$T_{\pm} \Psi_{s,t}^{(m,\epsilon)} = -\epsilon i \left(\frac{(t-s)(t+s)(t \pm 1)(m \mp t)(m \pm t + 1)}{t(t-s \pm 1)(t+s \pm 1)} \right)^{1/2} \Psi_{s, t \pm 1}^{(m,\epsilon)} \quad (5.13)$$

which, together with equation (3.20), give the action of the $so(2, 2)$ generators on the extended eigenfunctions of the Rosen–Morse potential. Here the conserved quantity m is related to the potential parameter C . The operators S_{\pm} change the other potential parameter B into $B' = B(s \pm 1)/s$, while leaving n fixed, while T_{\pm} change both B and n into $B' = B(t \pm 1)/t$ and $n' = n \mp \epsilon$, respectively.

6. The Kepler problem in a space of constant negative curvature

In a space of constant negative curvature $-R$, the radial wavefunction for an electron of mass μ in a Coulomb potential satisfies the equation [21]

$$\frac{d}{dx} \left(\sinh^2 x \frac{d\psi}{dx} \right) + [(\lambda - 2\nu) \sinh^2 x + 2\nu \sinh x \cosh x - l(l+1)]\psi = 0 \quad (6.1)$$

where $0 \leq x < \infty$, l is the angular momentum,

$$\nu \equiv \frac{ZR}{a_0} \quad \lambda \equiv \frac{2\mu R^2}{\hbar^2} E \quad (6.2)$$

and $a_0 = \hbar^2/(\mu e^2)$ denotes the Bohr radius. If $R \rightarrow \infty$ and $x \rightarrow 0$ in such a way that $xR \rightarrow r$, then equation (6.1) reduces to that of an electron in a central Coulomb field $-Ze^2/r$ in Euclidean space.

The negative energy eigenvalues and corresponding eigenfunctions are given by [21]

$$E_n = \frac{Ze^2}{R} - \frac{\hbar^2}{2\mu R^2} (n^2 - 1) - \frac{Z^2 e^4 \mu}{2\hbar^2 n^2} \quad n = 1, 2, \dots, n_{\max} \quad (6.3)$$

and

$$\psi_{n_r, l}(x) = N_{n_r, l} \sinh^l x e^{(n_r - \frac{\nu}{\hbar})x} {}_2F_1 \left(-n_r, l + 1 + \frac{\nu}{n}; 2l + 2; w \right) \quad w \equiv 1 - e^{-2x} \quad (6.4)$$

respectively. Here

$$n = n_r + l + 1 \quad (6.5)$$

where n_r is the radial quantum number, as in Euclidean space, but now n only takes a finite number of values n_{\max} . The latter corresponds to the number of independent functions (6.4) satisfying the normalization condition

$$\int_0^{\infty} dx \sinh^2 x |\psi_{n_r, l}(x)|^2 = 1 \quad (6.6)$$

for a given l value, and it is equal to the largest integer smaller than $\sqrt{\nu}$.

By setting

$$\psi(x) = \operatorname{cosech} x \phi(x) \quad (6.7)$$

equation (6.1) can be rewritten in a form similar to equation (3.30) with

$$p = 0 \quad \alpha = 1 \quad q = -\nu \quad (6.8)$$

and ψ, m, λ replaced by $\phi, l, \lambda - 1 - 2\nu$, respectively. The function $L(m)$ now becomes $L(l) = -l^2 - \nu^2/l^2$. It is an increasing function of l . The problem considered is therefore of class I. The counterpart of $m = 0, 1, \dots, l$ in the general theory of section 2 is $l = 0, 1, \dots, n - 1$, corresponding to $n_r = n - 1, n - 2, \dots, 0$.

From equations (3.18), (3.29) and (3.33) (where l is replaced by $n - 1$), we get

$$s = \frac{\nu}{n} \quad t = -n. \tag{6.9}$$

By using equations (6.4) and (6.5), and the results of [3], the corresponding extended eigenfunctions can be written as

$$\Psi_{s,t}^{(l)}(x, \xi, \eta) = (2\pi)^{-1} N_{s,t}^{(l)} e^{i(s\xi+t\eta)} \sinh^l x e^{-(s+t+l+1)x} \times {}_2F_1(t+l+1, s+l+1; 2l+2; w) \tag{6.10}$$

$$w \equiv 1 - e^{-2x}$$

where

$$N_{s,t}^{(l)} = \frac{2^{l+1}}{(2l+1)!} \left(\frac{(s+t)(s-t)(l-t)! \Gamma(s+l+1)}{(-t)(-t-l-1)! \Gamma(s-l)} \right)^{1/2}. \tag{6.11}$$

When acting on such extended eigenfunctions, the $su(1, 1)$ generators S_0, S_{\pm} of equations (3.19) and (3.31) become

$$\tilde{S}_0 = -i \frac{\partial}{\partial \xi} \quad \tilde{S}_{\pm} = e^{\pm i\xi} \left[\pm \sinh x \frac{\partial}{\partial x} + \cosh x \left(-i \frac{\partial}{\partial \xi} \pm 1 \right) - i \sinh x \frac{\partial}{\partial \eta} \right]. \tag{6.12}$$

The other $su(1, 1)$ generators T_0, T_{\pm} of equations (3.19) and (3.32) are similarly transformed into $\tilde{T}_0, \tilde{T}_{\pm}$, which can be obtained from equation (6.12) by the substitutions $\xi \leftrightarrow \eta, \partial/\partial \xi \leftrightarrow \partial/\partial \eta$.

After some calculations using well known properties of the hypergeometric function [20], we obtain

$$\tilde{S}_{\pm} \Psi_{s,t}^{(l)} = \left(\frac{(s+t)(s-t)(s \mp l)(s \pm l \pm 1)}{(s+t \pm 1)(s-t \pm 1)} \right)^{1/2} \Psi_{s \pm 1, t}^{(l)} \tag{6.13}$$

$$\tilde{T}_{\pm} \Psi_{s,t}^{(l)} = - \left(\frac{(s+t)(s-t)(-t \mp 1)(-t \pm l)(-t \mp l \mp 1)}{(-t)(s+t \pm 1)(s-t \mp 1)} \right)^{1/2} \Psi_{s, t \pm 1}^{(l)} \tag{6.14}$$

which, together with equation (3.20), give the action of the $so(2, 2)$ generators on the extended eigenfunctions of the Kepler problem. Here the conserved quantity is the angular momentum l . The operators \tilde{S}_{\pm} leave n (or n_r) unchanged, but change the potential parameter ν into $\nu' = \nu(s \pm 1)/s$, whereas \tilde{T}_{\pm} change both n (or n_r) and ν into $n' = n \mp 1$ (or $n'_r = n_r \mp 1$) and $\nu' = \nu(t \pm 1)/t$, respectively. From the definition of ν in equation (6.2), it is clear that \tilde{S}_{\pm} (resp. \tilde{T}_{\pm}) relate eigenfunctions of the Kepler problem in spaces of different curvature, R and $R' = R(s \pm 1)/s$ (resp. $R' = R(t \pm 1)/t$).

7. Conclusion

In this paper, we did show that the factorization method can be used in an effective way to construct satellite algebras for all the Hamiltonians that admit a type E factorization. Special emphasis was laid on the $so(2, 2)$ algebras characterizing hyperbolic potentials, but it is clear that a similar analysis could be carried out for the $so(2, 1)$ algebras appropriate to trigonometric potentials.

In the examples considered, we found that the conserved quantity, which is the eigenvalue of the satellite algebra Casimir operator, may have various physical meanings: a combination

of the potential parameters in the GMP case, one of the two potential parameters for the Rosen–Morse potential, and the angular momentum quantum number in the Kepler problem in a space of constant negative curvature. Similarly, the algebra generators may have various physical effects: relating eigenfunctions belonging to different potentials of the same family in the first two cases, or connecting eigenfunctions in spaces of different curvature in the last one. This hints at the existence of physical applications that may have been overlooked so far.

The approach used in this paper is not the only one allowing the construction of satellite algebras or, more generally, providing an algebraic treatment of the problems considered. Of particular significance is the work of Wu *et al* [5], who determined an $so(2, 2)$ algebra for the class of Natanzon potentials [22], which includes all the potentials solvable in terms of the hypergeometric or confluent hypergeometric function. A treatment of the same in terms of an $so(2, 1)$ algebra was also given by Cordero and Salamó [23]. It is worth mentioning too that the Kepler problem in a space of constant negative curvature was analysed in terms of a quadratic algebra [24]. We would like to stress, however, that our approach is the only one establishing a clear link with the factorization method of Schrödinger [2], and Infeld and Hull [3], and that in comparison with other papers we show more explicitly in the examples the effect of the action of the satellite algebra generators in terms of the potential parameters and the energy, which could be important from a physical viewpoint. A more detailed mathematical discussion of the irreducible representations will be given elsewhere.

As mentioned in section 1, considering type A and E factorizations is not a restriction as the other factorization types are but limiting cases of them. In forthcoming publications, we hope to come back to such limiting cases, as well as to a generalization of the factorization method recently proposed by Cariñena and Ramos [25].

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